COMPARISON OF OPTIMAL TARGET SELECTION RULES BASED ON DETERMINISTIC AND STOCHASTIC LANCHESTER MODELS OF HETEROGENEOUS COMBAT

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THESIS

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ABSTRACT

The simplest target selection problem in heterogeneous combat is the two-on-one battle. For a prescribed duration battle, deterministic and stochastic models using Lanchester's square law attrition mechanism are developed. Solutions of these models, obtained by the application of optimal control theory, are given, including the complete solution of the deterministic model and the optimal target selection rule and expected pay-off for special cases of the stochastic model. A numerical approximation for the general case of the stochastic model is obtained. Comparison of numerical results shows that the target selection rule specified by the deterministic models differs from the one given by the stochastic model.

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I. INTRODUCTION

One of the chief functions of mathematical models of combat is to provide insight for the development of tactics. In general, the assumptions needed to make the model tractable are such that the results cannot be directly applied. Frequently, however, the results reveal a structure which is obscured by the complexity of the real world problem.

There are a number of different approaches to combat attrition modeling, each with its own advantages and disadvantages. Generally speaking, greater reality is introduced into such models at the expense of increased complexity.

A factor to be considered is whether the results are influenced by the approach used. It may be that for a particular type of problem, a complex, difficult approach yields nothing that could not be gained from a simpler model.

One of the first mathematical models of combat was presented by Rear Admiral Bradley A. Fiske, USN, in 1905 [1]. Fiske considered an engagement between two fleets. He assigned a numerical value representing strength to each fleet and assumed that the damage a fleet could inflict was proportional to its strength. In a series of tables, he showed that the value of the weaker fleet

in relation to the stronger fleet decreased at an accelerating rate as the battle progressed. Fiske's logical development and conclusions are the same as those later formalized by Lanchester.

Pioneering work on combat attrition modelling was done by F. W. Lanchester [2] in the early nineteen hundreds. He postulated that the combat attrition process could be described by a system of ordinary differential equations.

Lanchester considered two types of combat: individual combat as represented by ancient combat; and modern combat characterized by the introduction of weapons with large firepower.

Letting x(t) and y(t) represent the opposing force levels, in the first case, the attrition depends on both force levels and is given by

$$\frac{dx}{dt} = -ax(t)y(t) \tag{1}$$

$$\frac{dy}{dt} = -bx(t)y(t) \tag{2}$$

where a and b are called Lanchester attrition-rate coefficients. Dividing Eq. 1 by Eq. 2 and integrating yields the force level equation:

$$b\left(x(0) - x(t)\right) = a\left(y(0) - y(t)\right)$$

which is known as the Lanchester linear law. If x is to win $\left(x(t) > 0,\ y(t) = 0\right)$, then the initial force levels must satisfy

$$x(0) > \frac{a}{b} y(0)$$

In the second case, the attrition depends only on the opposing force level and is given by

$$\frac{dx}{dt} = -ay(t)$$

$$\frac{dy}{dt} = -bx(t)$$

From this system of equations, Lanchester's square law,

$$b\left(x^2(0) - x^2(t)\right) = a\left(y^2(0) - y^2(t)\right)$$

is derived with the condition for x to win that

$$x^2(0) > \frac{a}{b} y^2(0).$$

Since Lanchester's work, numerous extensions, modifications, etc., have been advanced. A survey of the work in this area can be found in Dolansky [3].

An obvious drawback to the Lanchester formulations is that they are deterministic. Given the attrition coefficient, the winner is determined by the initial force levels. Clearly, this is not true in actual combat. There are a great many other factors, which may be considered as random variables, which enter into the determination. Apparently the first stochastic formulation was given by Koopman [4,5].

Weiss [6] noted a number of deficiencies. Of particular significance is the fact that the Lanchester model assumes that all units on a side are of the same type (homogeneous

forces). In actual combat, there are units of many different types on each side (heterogeneous forces).

The simplest heterogeneous force model involves a single force, Y, facing an opposing force composed of two types of units, X_1 and X_2. The attrition structure is described by some function of attrition coefficients and force levels. The Y force commander faces an allocation of fire problem. How should he divide his fire between X_1 and X_2?

In the following, deterministic and stochastic models of this problem are developed and the solution obtained by application of optimal control theory is given. For particular numerical values, the optimal allocation of fire and the terminal pay-off are compared.

Table I. Notation

$x_1(t)$, $x_2(t)$, $y(t)$ -- State variables, number of survivors of X_1, X_2, Y at time t.

T -- Specified maximum duration of the battle.

p, q, r -- Value of a survivor of the battle of X_1, X_2, Y respectively.

a_1 -- Attrition coefficient, rate at which one y attrits X_1.

a_2 -- Rate at which one y attrits X_2.

b_1 -- Rate at which one x_1 attrits Y.

b_2 -- Rate at which one x_2 attrits Y.

ϕ -- Decision variable, fraction of Y fire directed at X_1, $0 \leq \phi \leq 1$.

$$R = \frac{a_1 b_1}{a_2 b_2}$$

$$\delta = \frac{a_1 p}{a_2 q}$$

$$s = b_1 x_1(0) + b_2 x_2(0)$$

$$Z = \frac{R - s}{R - 1}$$

$$A = \frac{R(Z^2 - 1) - Z^2}{(Z - 1)^2}$$

$$B = \frac{R(Z^2 - 1) - Z^2}{Z^2}$$

$$\alpha = \frac{r}{q}\sqrt{\frac{b_2}{a_2}}$$

II. THE DETERMINISTIC MODEL

The development of the model requires three major decisions. An objective function must be chosen. A stopping rule must be defined and the structure of the attrition mechanism selected.

Among the Y force commander's objectives might be to maximize casualties inflicted, minimize the casualties incurred or to maximize the duration of the battle. The objective function used in this model is

$$\max_{\phi} \; ry(T) - px_1(T) - qx_2(T)$$

that is, the Y force commander desires to allocate his forces to maximize the net worth of the survivors, the terminal pay-off. Appropriate choices of the weighting factors, p, q, and r, reflect Y's feeling regarding casualties. At one extreme, p and q set equal to zero correspond to minimizing casualties incurred, while r set equal to zero essentially maximizes casualties inflicted. It can be shown that maximizing the duration of the battle leads to a different allocation policy.

There are three ways in which the battle may end. One force may decide to disengage, either as a function of casualties incurred or after some specified time, or one force may be driven to zero. In the deterministic formulation, there is a one-to-one correspondence between casualties and elapsed time.

The following stopping rule is used in this model. With T specified, there are four states in which the battle can end:

C_1 $y(T) > 0$ and either $x_1(T) > 0$ or $x_2(T) > 0$ or both (time expires).

C_2 $y(t) = 0$ while either $x_1(t) > 0$ or $x_2(t) > 0$ or both for $0 \leq t \leq T$ (Y loses).

C_3 $x_1(t) = x_2(t) = 0$, $y(t) > 0$ for $0 \leq t \leq T$ (X loses).

C_4 $x_1(t) = x_2(t) = y(t) = 0$ for $0 \leq t \leq T$ (draw).

States C_2 to C_4 are referred to as fight-to-the-finish or terminal control battles, while C_1 is a prescribed duration battle. The model is formulated as a prescribed duration battle with consideration given to premature terminations (C_2, C_3, or C_4). If T is large enough, the premature terminations will determine the actual time the battle ends.

The attrition structure used is the Lanchester square law. These choices lead to the following model.

$$\max_{0 \leq \phi \leq 1} \; ry(T) - px_1(T) - qx_2(T) \tag{3}$$

$$\text{ST} \quad \frac{dx_1}{dt} = -\phi a_1 y(t)$$

$$\frac{dx_2}{dt} = -(1-\phi)\, a_2 y(t) \tag{4}$$

$$\frac{dy}{dt} = -\left(b_1 x_1(t) + b_2 x_2(t)\right)$$

$x_1, x_2, y \geq 0$. T specified.

A number of assumptions are involved in the formulation and use of the model.

A1. Each unit has perfect information regarding the state and location of enemy units.

A2. Fire may be shifted instantaneously.

A3. Fire is uniformly distributed over surviving units.

A4. All enemy units are within range.

A5. The effects of successive rounds are independent.

A6. The attrition coefficients a_1, a_2, b_1, b_2 are constant throughout the battle.

A7. Fractional casualties are allowed.

A8. $a_1 b_1 \geq a_2 b_2$. Since the labelling of forces is arbitrary, this is a nonrestrictive assumption.

A1 through A5 are inherent in the Lanchester square law formulation, as developed by Barfoot [7].

This problem was first formulated and partially solved by Isbell and Marlow [8], before Pontryagin announced his maximum principle. Using subsequent developments in modern optimal control theory, Taylor [9, 10] has obtained a complete solution to both the fixed-duration and the fight-to-the-finish battle.

III. SOLUTION TO THE DETERMINISTIC MODEL

The derivation of Taylor's solution is exceedingly lengthy and tedious. The results are summarized in Appendix A. This section discusses Taylor's solution technique and the insight into the structure of the optimal allocation policy provided by this solution.

The solution procedure begins by forming the Hamiltonian

$$H(t, x_i, p_i, \phi) = -\{p_1(t)\,\phi a_1 y + p_2(t)(1-\phi)\,a_2 y + p_3(t)(b_1 x_1 + b_2 x_2)\} \tag{5}$$

where p_1, p_2, and p_3 are the dual variables associated with X_1, X_2 and Y respectively.

By the Pontryagin maximum principle[1], maximizing Eq. 5 at all points in time yields the maximum of Eq. 3. Rewriting Eq. 5 as

$$H = -\{\phi\left(a_1 p_1(t) - a_2 p_2(t)\right) y + a_2 p_2(t)\, y + p_3(t)(b_1 x_1 + b_2 x_2)$$

it is clear that the optimal control, ϕ^* is given by

$$\phi^* = \begin{cases} 1 \text{ if } a_2 p_2(t) > a_1 p_1(t) \\ 0 \text{ if } a_2 p_2(t) < a_1 p_1(t) \end{cases} \tag{6}$$

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[1]This paper uses the American form of the maximum principle [11]. Pontryagin, et. al., [12] use an equivalent form which differs in sign conventions.

It can be shown that $a_2p_2 = a_1p_1$ cannot hold over a finite interval of time, hence the optimal control is to always concentrate fire against one target type.

The adjoint system of differential equations for the dual variables is given by

$$\frac{dp_1}{dt} = b_1p_3(t)$$

$$\frac{dp_2}{dt} = b_2p_3(t) \tag{7}$$

$$\frac{dp_3}{dt} = p_1(t)a_1\phi + p_2(t)a_2(1-\phi)$$

When the optimal value function is differentiable, the dual variables represent the change in the optimal value function caused by a change in the corresponding state variable. In particular, if no force has been driven to zero at the end of the battle, the final values of the dual variables are easily found. Differentiating Eq. 3 with respect to the state variables yields

$$p_1(T) = -p$$

$$p_2(T) = -q$$

$$p_3(T) = r.$$

If either X_1 or X_2 is driven to zero during the course of the battle, the final values of the dual variables can no longer be determined as above; instead, the theory of state variable inequality constraints must be applied.

Taylor [Appendix E of 10] presents a detailed discussion of this theory.

When a force is driven to zero, the optimal trajectory is said to enter a constraint sub-arc. The point of entry is called a corner. In this problem, a nonnegativity constraint is binding on a constrained sub-arc. One treatment, developed by Gamkrelidze [12] is to modify the Hamiltonian by adjoining the first time derivative of the constraints to the criterion functional with additional Lagrange multipliers.

$$H = - p_1 a_1 \phi y - p_2 a_2 (1-\phi) y - p_3 (b_1 x_1 + b_2 x_2)$$
$$- \mu_1(t) a_1 \phi y - \mu_2(t) a_2 (1-\phi) y$$

The μ_i's are multipliers satisfying

$$\mu_i \begin{cases} = 0 & x_i > 0 \\ \geq 0 & x_i = 0 \end{cases} \qquad i = 1, 2$$

as well as Gamkrelidze's condition

$$\frac{d\mu_i}{dt} \leq 0 \qquad \text{for } x_i = 0$$

There are also corner conditions on the dual variables at an entrance to a constrained sub-arc. If X_1 is driven to zero at t_1, these are

$$p_1(t_1^-) = p_1(t_1^+) + \mu_1(t_1^+)$$
$$p_2(t_1^-) = p_2(t_1^+)$$
$$p_3(t_1^-) = p_3(t_1^+)$$

In addition, the Hamiltonian is continuous across the corner. Approaching the corner (t_1^-), $\phi=1$, $p_3 b_1 x_1$ approaches zero, and $\mu_1=0$. After the corner (t_1^+), $\phi=0$, $x_1=0$, and $\mu_2=0$. From the continuity of the Hamiltonian

$$H(t_1^-) = -p_1(t_1^-)a_1 y - p_3 b_2 x_2$$

$$= -p_2(t_1^+)a_2 y - p_3 b_2 x_2 = H(t_1^+)$$

therefore

$$p_1(t_1^-) = \frac{a_2}{a_1}\, p_2(t_1^+) = \frac{a_2}{a_1}\, p_2(t_1^-).$$

On a constrained sub-arc with $x_1=0$, the multiplier is determined from the condition

$$\frac{dH^*}{d\phi} = 0$$

which yields

$$\mu_1(t) = \frac{1}{a_1}\left(a_2 p_2(t) - a_1 p_1(t)\right).$$

Since

$$\frac{du_1}{dt} = \frac{1}{a_1}\left(a_2\frac{dp_2}{dt} - a_1\frac{dp_1}{dt}\right)$$

$$= \frac{p_3(t)}{a_1}\,(a_2 b_2 - a_1 b_1)$$

using the adjoint equations (Eq. 7) it is clear that Gamkredlidze's condition is only satisfied on the constrained sub-arc $x_1=0$ if $a_1 b_1 \geq a_2 b_2$ since it is readily shown that $p_3(t) \geq 0$.

A similar argument shows that Gamkrelidze's condition is satisfied on the constrained sub-arc $x_2=0$ only if $a_1b_1 < a_2b_2$ which contradicts A8. Therefore, it is optimal to drive X_2 to zero only at the end of the battle, if at all.

Since the values of the dual variables are known for t=T, it is convenient to define the backwards time variable

$$\tau = T - t \qquad 0 \le t \le T$$

and so

$$\frac{dp_i}{d\tau} = \frac{dp_i}{dt}\frac{dt}{d\tau} = -\frac{dp_i}{dt} \qquad i = 1, 2, 3 \qquad (8)$$

Substitution of Eq. 8 into Eq. 7 yields a system of differential equations with known initial conditions which can be solved for the dual variables as a function of time.

Once the dual variable functions are found, the optimal control is given by Eq. 6. The time at which fire is switched from X_1 to X_2 is found from

$$a_1p_1(t) = a_2p_2(t).$$

The trajectories of the state variables are easily determined. For an interval (t_1, t_2) where $\phi=1$, Eq. 4 becomes

$$\frac{dx_1}{dt} = -a_1y(t) \qquad x_1(t_1) \text{ given}$$

$$\frac{dx_2}{dt} = 0 \qquad x_2(t_1) \text{ given}$$

$$\frac{dy}{dt} = -\left[b_1x_1(t) + b_2x_2(t)\right] \qquad y(t_1) \text{ given}$$

which can be solved to get for $t_1 \leq t \leq t_2$

$$x_1(t) = x_1(t_1) + a_1\left\{\frac{\left(b_1x_1(t_1) + b_2x_2(t_2)\right)}{a_1b_1} \cosh (\sqrt{a_1b_1}t)\right.$$

$$\left. - \frac{y(t_1)}{\sqrt{a_1b_1}} \sinh (\sqrt{a_1b_1}t) - \frac{\left(b_1x_1(t_1) + b_2x_2(t_1)\right)}{a_1b_1}\right\}$$

$$x_2(t) = x_2(t_1) \tag{9}$$

$$y(t) = y(t_1) \cosh (\sqrt{a_1b_1}t)$$

$$- \frac{\left(b_1x_1(t_1) + b_2x_2(t_1)\right)}{\sqrt{a_1b_1}} \sinh (\sqrt{a_1b_1}t).$$

Similarly, for an interval (t_3, t_4) where $\phi = 0$, Eq. 4 becomes

$$\frac{dx_1}{dt} = 0 \qquad x_1(t_3) \text{ given}$$

$$\frac{dx_2}{dt} = - a_2y(t) \qquad x_2(t_3) \text{ given}$$

$$\frac{dy}{dt} = - \left(b_1x_1(t) + b_2x_2(t)\right) \qquad y(t_3) \text{ given}$$

which gives for $t_3 \leq t \leq t_4$

$$x_1(t) = x_1(t_3)$$

$$x_2(t) = x_2(t_3) + a_2\left\{\frac{\left(b_1x_1(t_3) + b_2x_2(t_3)\right)}{a_2b_2} \cosh (\sqrt{a_2b_2}t)\right.$$

$$\left. - \frac{y(t_3)}{\sqrt{a_2b_2}} \sinh (\sqrt{a_2b_2}\, t) - \frac{\left(b_1x_1(t_3) + b_2x_2(t_3)\right)}{a_2b_2}\right\}$$

$$y(t) = y(t_3) \cosh (\sqrt{a_2 b_2}\, t)$$

$$- \frac{(b_1 x_1(t_3) + b_2 x_2(t_3))}{\sqrt{a_2 b_2}} \sinh (\sqrt{a_2 b_2}\, t) \quad . \qquad (10)$$

Neglecting drawn battles (terminal state C_4,) the terminal states can be partitioned as shown in Table II. Using the optimal control determined by Eq. 6, the state equations, Eqs. 9 and 10 can be integrated in forward time to determine what combinations of parameters and initial force levels can lead to particular terminal states. These conditions define the domain of controllability.

Two additional items must be checked. First, the domains of controllability must cover the entire state space. If not, then there exists a singular surface where the Pontryagin maximal principle does not apply and other methods must be used to determine the optimal control. Such a surface is not present in this problem.

Secondly, it may be that the domains of controllability overlap. If this is the case, then the optimal control is the one leading to the terminal state with the largest pay-off. Unless one terminal state dominates the other, the region of overlap is partitioned by a dispersal surface into subregions from which optimal extremals lead to a particular terminal state.

Dispersal surfaces are present in this case. The resolution of these areas can be found in Taylor [10]. As this paper is primarily concerned with numerical results,

Table II. Terminal States

State	Y	X_1	X_2	T_1
S1	>0	>0	>0	$=T$
S2	>0	$=0$	>0	$=T$
S3	>0	>0	$=0$	$=T$
S4	$=0$	>0	>0	$\leq T$
S5	$=0$	$=0$	>0	$\leq T$
S6	$=0$	>0	$=0$	$\leq T$
S7[1]	>0	$=0$	$=0$	$\leq T$
S8[2]	>0	$=0$	$=0$	$\leq T$

T : Specified duration of the battle.

T_1: Time battle actually ends.

S1 - S3 partition C_1 (fixed-duration battle)

S4 - S6 partition C_2 (Y loses)

S7 - S8 partition C_3 (X loses)

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[1] X_1 destroyed first.

[2] X_2 destroyed first.

the question is resolved by computing the pay-off for each possible terminal state and choosing the maximum.

Insight into the allocation policy can be gained by considering the interpretation of the quantities a_1b_1, a_2b_2, a_1p, and a_2q. The quantity $a_1b_1(a_2b_2)$ is a strategic or long term measure. It can be interpreted as a measure of the rate at which Y destroys X_1's (X_2's) ability to

destroy Y. On the other hand, $a_1p(a_2q)$ is a tactical or short term measure, interpreted as the rate at which Y destroys X_1's (X_2's) value. By A8, $a_1b_1 \geq a_2b_2$, so there are two cases to consider.

If $a_1p \geq a_2q$, the situation is relatively simple. Y receives more return in both the strategic and tactical sense by firing at X_1, hence it is never optimal to fire at X_2 unless $X_1 = 0$. Accordingly, in this case, no extremals lead to states S3, S6, or S8. For all other terminal states, $\phi^*=1$ until $X_1=0$ or $Y=0$.

Things are more complicated if $a_1p < a_2q$, since Y obtains a better tactical return by firing at X_2, but a better strategic return by firing at X_1. At the termination of the battle, only the tactical return need be considered, so clearly, $\phi^*(T)=0$ as long as there are any X_2 survivors at T.

However, since Y is more effective in destroying X_1's kill capability than X_2's, it is in Y's best interest to fire at X_1 until there is just enough time remaining to destroy X_2 then shift fire. Otherwise, Y will incur needless casualties.

IV. THE STOCHASTIC MODEL

A. STOCHASTIC CONTROL PROBLEMS

The addition of stochastic elements to the optimal control problem introduces several complications. There are a number of aspects which must be considered, since different interpretations lead to different courses of action. These are discussed at length by Dreyfus [13].

The state of knowledge of the underlying stochastic process must be considered. In the simplest case, the properties of the process may be known. On the other hand, if the nature of the process is not known, a learning element enters the problem. The course of action chosen must be refined as the process evolves and knowledge is gained. This is known as an adaptive control problem.

Another important factor is the ability to observe the state of the system. In the simplest case, the state can be observed precisely at any particular point. If the state is only partially observable, then an estimation problem must also be treated.

Since the trajectory is no longer precisely determined, the objective function becomes a random variable. Among the alternatives for treating the objective function are: the maximization of an expected value; maximizing the probability of reaching a particular state; or minimizing the variance of the terminal state.

A significant difference from the deterministic problem is that the manner in which the control is applied leads to different results. There are three basic control policies to be considered: open loop; feedback; and open loop-feedback.

An open loop control policy represents the case in which the control must be specified before the process starts and cannot be altered. In a process evolving through time, the control is expressed as a function of time, given the initial state.

A feedback or close loop policy assumes the ability to continuously observe the state of the system and modify the control. Mathematically, the control is expressed as a function of time and state, given the initial state.

The open loop-feedback policy arises if it is impossible to continuously observe the system. Generally, the system can be observed only at particular points in time. In this case, an open loop control based on the initial state is applied to start the process. After an interval of time, the state is observed and another open loop control is applied, based on the observed state. This continues until the process terminates.

It is fairly obvious that the feedback policy is the "best" of the three. The process can be rigidly controlled since, should the process start to behave in a nonoptimal fashion, it can be corrected immediately. In an open loop situation, a more delicate touch is required. There is no

way to correct any over-control. The open loop-feedback corresponds to the physical situation usually faced in the actual problem, as opposed to the model.

In the following section, a feedback control will be found for a model where the underlying process is known and the states are fully observable.

B. EXACT SOLUTION OF THE STOCHASTIC MODEL

This model differs from the deterministic model in the structure of the attrition mechanism. The physical situation is the same as in Section II. Assumptions A1 through A6 and A8 are retained. A7 is replaced by

A7'. Only integer casualties can occur and Y can allocate only integer numbers of troops.

The objective function is

$$\max_{\phi} E\{ry(T) - px_1(T) - qx_2(T)\}$$

This model treats the prescribed duration battle. For reasons similar to those of the deterministic model, it is convenient to work in backwards time, $\tau = T-t$ where T is the prescribed termination time.

Instead of a deterministic square law attrition mechanism, the attrition is modelled as a stationery Markov process. The following probability statements are assumed for a sufficiently small increment of time $\Delta\tau$:

A9. Prob [one X_1 casualty in $\Delta\tau$] = $a_1 y\phi\Delta\tau$

Prob [more than one X_1 casualty in $\Delta\tau$] = $o(\Delta\tau)$

Prob [one X_2 casualty in $\Delta\tau$] = $(1-\phi)\ a_2 y\Delta\tau$

Prob [more than one X_2 casualty in $\Delta\tau$] = $o(\Delta\tau)$

Prob [one Y casualty in $\Delta\tau$] = $(b_1x_1 + b_2x_2)\Delta\tau$

Prob [more than one Y casualty in $\Delta\tau$] = $o(\Delta\tau)$

Prob [more than one casualty of any type in $\Delta\tau$]

$= o(\Delta\tau)$

where $o(\Delta\tau)$ is a quantity such that $\lim_{\Delta\tau\to 0} \frac{o(\Delta\tau)}{\Delta\tau} = 0$. Given the above, it follows that

$$P\,[\text{no } X_1 \text{ casualties in } \Delta\tau] = 1 - \phi a_1 y\Delta\tau + o(\Delta\tau)$$

and similarly for X_2 and Y.

The assumption that Y has perfect information about the state of the system at any time and the ability to instantaneously shift fire leads to a feedback or closed loop optimal control.

To solve this model, the optimal expected value function, S, is defined as

$$S[\tau, x_1(\tau), x_2(\tau), y(\tau)] = E[ry(0) - px_1(0) - qx_2(0)]$$

given that the state at time τ is $x_1(\tau)$, $x_2(\tau)$ and $y(\tau)$ and that an optimal control has been applied in the interval $[0,\tau]$.

Table III lists five events which could take place in an interval of length $\Delta\tau$, $[\tau, \tau-\Delta\tau]$ and their probability of occurrence. An expression for $S(\tau, x_1, x_2, y)$ is developed by the application of Bellman's Principle of Optimality [14] which states:

Table III. Transition Probabilities

State @τ	Casualties	Probability
x_1, x_2, y	none	$1 - \{\phi a_1 y + (1-\phi)a_2 y + b_1 x_1 + b_2 x_2\}\Delta\tau$
x_1-1, x_2, y	one X_1	$\phi a_1 y \Delta\tau$
x_1, x_2-1, y	one X_2	$(1-\phi)a_2 y\Delta\tau$
x_1, x_2, $y-1$	one Y	$(b_1 x_1 + b_2 x_2)\Delta\tau$
Various	more than 1	$o(\Delta\tau)$

> An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

By definition, the return from any of the states listed in Table III resulting from following an optimal policy to the end of the battle is $S(\tau-\Delta\tau,\ x_1,\ x_2,\ y)$. Therefore, the optimal value of ϕ in the interval $[\tau,\ \tau-\Delta\tau]$ is the one which maximizes the return expected, starting from state x_1, x_2, y at time τ. This can be expressed as

$$\begin{aligned} S[\tau,\ x_1,\ x_2,\ y] = \max_{0\leq\phi\leq 1} \{&[1 - \{\phi a_1 y + (1-\phi)a_2 y \\ &+ b_1 x_1 + b_2 x_2\}\Delta\tau]S(\tau-\Delta\tau, x_1, x_2, y) \\ &+ \phi a_1 y\Delta\tau S(\tau-\Delta\tau, x_1-1, x_2, y) \\ &+ (1-\phi)a_2 y\Delta\tau S(\tau-\Delta\tau, x_1, x_2-1, y) \\ &+ (b_1 x_1 + b_2 x_2)\Delta\tau S(\tau-\Delta\tau, x_1, x_2, y-1) + o(\Delta\tau)\} \end{aligned} \tag{11}$$

Since $S(\tau-\Delta\tau, x_1, x_2, y)$ does not depend on the control variable, ϕ, this can be rewritten as

$$\frac{S(\tau, x_1, x_2, y) - S(\tau-\Delta\tau, x_1, x_2, y)}{\Delta\tau}$$

$$= \max_{0\leq\phi\leq 1} \{(b_1x_1 + b_2x_2)[S(\tau-\Delta\tau, x_1, x_2, y-1)$$

$$-S(\tau-\Delta\tau, x_1, x_2, y)]$$

$$+ y\{\phi a_1[S(\tau-\Delta\tau, x_1-1, x_2, y) - S(\tau-\Delta\tau, x_1, x_2, y)]$$

$$+ (1-\phi)a_2[S(\tau-\Delta\tau, x_1, x_2-1, y) - S(\tau-\Delta\tau, x_1, x_2, y)]$$

$$+ \frac{o(\Delta\tau)}{\Delta\tau} \}$$

In the limit as $\Delta\tau\to 0$,

$$\frac{dS(\tau, x_1, x_2, y)}{d\tau} = (b_1x_1 + b_2x_2)[S(\tau, x_1, x_2, y-1)$$

$$- S(\tau, x_1, x_2, y)]$$

$$+ y \max_{0\leq\phi\leq 1} \{\phi(a_1[S(\tau, x_1-1, x_2, y) - S(\tau, x_1, x_2, y)]$$

$$- a_2[S(\tau, x_1, x_2-1, y) - S(\tau, x_1, x_2, y)]) \quad (12)$$

$$+ a_2[S(\tau, x_1, x_2-1, y) - S(\tau, x_1, x_2, y)]$$

Equation 12 can also be obtained directly from Kushner's general result on the optimal control of a "Poisson" process [15].[2]

To determine ϕ, the switching function

$$w(\tau) = a_1\{S(\tau, x_1-1, x_2, y) - S(\tau, x_1, x_2, y)\}$$
$$-a_2\{S(\tau, x_1, x_2-1, y) - S(\tau, x_1, x_2, y)\} \quad (13)$$

is used. Equation 12 then becomes

$$\frac{dS(\tau, x_1, x_2, y)}{d\tau} = (b_1x_1 + b_2x_2)\{S(\tau, x_1, x_2, y-1)$$
$$- S(\tau, x_1, x_2, y)\}$$
$$+ a_2y\{S(\tau, x_1, x_2-1, y) - S(\tau, x_1, x_2, y)\}$$
$$+ y \max_{0\le\phi\le 1} \{\phi w(\tau)\}$$

Clearly, the optimal control is given by

$$\phi^*(\tau) = \begin{cases} 1 & w(\tau) \ge 0 \\ 0 & w(\tau) < 0 \end{cases}$$

In the deterministic case where the optimal control depended on the dual variables, it was necessary to treat the case where a force is driven to zero separately since

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[2]This contains a typographical error. The unnumbered equation after Eq. 7 should read

$$0 = \min_u \{k + V_t + \sum_1^n p_i g_i + \sum_1^n c_i [V(x+\lambda_i, t) - V(x, t)]\}$$

If i=1, this reduces to Dreyfus result [13].

the boundary influenced the solution for the dual variables. In the stochastic case, the optimal control depends only on the optimal expected value function. As shown below, the initial conditions for S are easily determined even when a force is driven to zero. Therefore, no special distinction between the fixed duration and the fight to the finish battles is necessary to solve for the optimal expected value function.

If $\tau=0$ (i.e. at the end of the battle), then

$$S(0, x_1, x_2, y) = ry - px_1 - qx_2 \tag{14}$$

for all values x_1, x_2, and y.

If $y=0$, there is no battle and

$$S(\tau, x_1, x_2, 0) = -(px_1 + qx_2) \tag{15}$$

for all values of τ, x_1, and x_2.

Similarly, if $x_1 = x_2 = 0$, there is no battle and

$$S(\tau, 0, 0, y) = ry \tag{16}$$

for all values of τ and y.

If $x_1 = 0$, then clearly, $\phi = 0$ for all τ and Eq. 12 reduces to

$$\frac{dS(\tau, 0, x_2, y)}{d\tau} = b_2x_2[S(\tau, 0, x_2, y-1) - S(\tau, 0, x_2y)]$$

$$+ ya_2[S(\tau, 0, x_2-1, y) - S(\tau, 0, x_2, y)]$$

with the initial condition from Eq. 14

$$S(0, 0, x_2, y) = ry - qx_2.$$

In particular, for $x_2 = y = 1$

$$\frac{dS(\tau, 0, 1, 1)}{d\tau} = b_2[S(\tau, 0, 1, 0) - S(\tau, 0, 1, 1)]$$

$$+ a_2[S(\tau, 0, 0, 1) - S(\tau, 0, 1, 1)]$$

$$= b_2[-q-S(\tau, 0, 1, 1)] + a_2[r-S(\tau,0,1,1)]$$

using Eqs. 15 and 16 or

$$\frac{dS(\tau, 0, 1, 1)}{d\tau} = -(b_2 + a_2)S(\tau, 0, 1, 1) + a_2 r - b_2 q$$

with $S(0, 0, 1, 1) = r - q$. This is an ordinary differential equation of the form

$$\frac{dx}{dt} = c_1 x + c_2$$

which is easily solved to give

$$S(\tau, 0, 1, 1) = \left(\frac{b_1 r - a_2 q}{a_2 + b_2}\right) \exp\left\{-(a_2 + b_2)\tau\right\}$$

$$+ \left(\frac{a_2 r - b_2 q}{a_2 + b_2}\right) . \qquad (17)$$

Similarly, if $x_2 = 0$, then $\phi = 1$ for all τ and for $x_1 = y = 1$, Eq. 12 reduces to

$$\frac{dS(\tau, 1, 0, 1)}{d\tau} = b_1\{-p-S(\tau, 1, 0, 1)\} + a_1\{r-S(\tau,1,0,1)\}$$

$$= -(b_1 + a_1)S(\tau, 1, 0, 1) + a_1 r - b_1 p$$

with $S(0, 1, 0, 1) = r - p$. The solution in this case is

$$S(\tau,\ 1,\ 0,\ 1) = \left(\frac{b_1 r - a_1 p}{a_1 + b_1}\right) \exp\{-(a_1 + b_1)\tau\}$$

$$+ \left(\frac{a_1 r - b_1 p}{a_1 + b_1}\right) . \qquad (18)$$

Letting $x_1 = x_2 = y = 1$, Eq. 12 becomes

$$\frac{dS(\tau,\ 1,\ 1,\ 1)}{d\tau} = (b_1 + b_2)[S(\tau,\ 1,\ 1,\ 0) - S(\tau,\ 1,\ 1,1)]$$

$$+ a_2[S(\tau,\ 1,\ 0,\ 1) - S(\tau,\ 1,\ 1,\ 1)]$$

$$+ \max_{0 \le \phi \le 1} \{\phi(a_1[S(\tau,\ 0,\ 1,\ 1) - S(\tau,\ 1,\ 1,\ 1)]$$

$$- a_2[S(\tau,\ 1,\ 0,\ 1) - S(\tau,\ 1,\ 1,\ 1)])\} . \qquad (19)$$

The switching function determines ϕ. At the end of the battle

$$w(0) = a_1(r - q - r + p + q) - a_2(r - p - r + p + q)$$

$$= a_1 p - a_2 q$$

so there are two cases to consider.

1. Case I

$$a_1 p \ge a_2 q$$

$\phi^*(\tau) = 1$ for $0 \le \tau \le \tau_1$ where τ_1 is the time when Y switches fire from X_1 to X_2, determined by

$$w(\tau_1) = 0.$$

This is consistent with the interpretation of $a_1 p$ as a tactical measure. Certainly, at the end of the battle,

Y should direct his fire to get the greater return. In this case, Eq. 19 becomes

$$\frac{dS(\tau, 1, 1, 1)}{d\tau} = (b_1 + b_2)\{S(\tau, 1, 1, 0) - S(\tau, 1,1,1)\}$$

$$+ a_1\{S(\tau, 0, 1, 1) - S(\tau, 1, 1, 1)\}$$

with the initial condition

$$S(0, 1, 1, 1) = r - p - q.$$

Substituting Eqs. 15 and 17, the solution is given by

$$S(\tau, 1, 1, 1) = \frac{a_1(b_2r - a_2q)}{(a_1 + b_1 - a_2)(a_2 + b_2)} \exp\{-(a_2+b_2)\tau\}$$

$$+ \left\{\frac{[(b_1 - a_2)(b_1 + b_2) + a_1b_1]r}{(a_1 + b_1 - a_2)(a_1 + b_1 + b_2)} - \frac{a_1p}{(a_1 + b_1 + b_2)}\right.$$

$$\left. + \frac{a_1a_2q}{(a_1 + b_1 - a_2)(a_1 + b_1 + b_2)}\right\}$$

$$\exp\{-(a_1 + b_1 + b_2)\tau\} + \left\{\frac{a_1a_2r}{(a_2 + b_2)(a_3 + b_1 + b_2)}\right.$$

$$\left. - \frac{(b_1 + b_2)p}{(a_1 + b_1 + b_2)} - \frac{[(b_1 + b_2)(a_2 + b_2) + a_1b_2]q}{(a_2 + b_2)(a_1 + b_1 + b_2)}\right\}$$

for $0 \leq \tau \leq \tau_1$.

Using this in Eq. 13, the switching time τ_1 is determined. Depending on the value of T, it may be that $\tau_1 > T$ and so no switch occurs during the battle. If $\tau_1 < T$, then in the interval $\tau_1 \leq \tau \leq \tau_2$ where τ_2 is the time of switch from X_2 back to X_1, $\{w(\tau_2) = 0\}$, $\phi^*(\tau) = 0$ and

$$S(\tau, 1, 1, 1) = \frac{a_2(b_1 r - a_1 p)}{(a_2 + b_2 - a_1)(a_1 + b_1)} \{ \exp\{-(a_1+b_1)\tau\}$$

$$- \exp \{ (a_2 + b_2)(\tau_1 - \tau) - a_1\tau_1 - b_1\}\}$$

$$+ \left\{ S_0 - \frac{a_1 a_2 r}{(a_1 + b_1)(a_2 + b_2 + b_1)} \right.$$

$$\left. + \frac{[(b_1 + b_2)(a_1 + b_1) + a_2 b_2]p}{(a_1 + b_1)(a_2 + b_2 + b_1)} + \frac{(b_1 + b_2)q}{(a_2 + b_2 + b_1)} \right\}$$

$$\exp \{-(a_2 + b_2 + b_1)(\tau - \tau_1)\} + \left\{ \frac{a_1 a_2 r}{(a_1+b_1)(a_2+b_2+b_1)} \right.$$

$$\left. - \frac{[(b_1 + b_2)(a_1 + b_1) + a_2 b_1]p}{(a_1 + b_1)(a_2 + b_2 + b_1)} - \frac{(b_1 + b_2)q}{(a_2 + b_2 + b_1)} \right\}$$

where $S_0 = S(\tau_1, 1, 1, 1)$.

2. <u>Case II</u>

$a_2 q > a_1 p$

In this case, $\phi(\tau) = 0$ for $0 \leq \tau \leq \tau_1$ where τ_1 is the time that fire switches from X_2 to X_1. For this interval

$$S(\tau, 1, 1, 1) = \frac{a_2(b_1 r - a_1 p)}{(a_2 + b_2 - a_1)(a_1 + b_1)} \exp \{-(a_1+b_1)\tau\}$$

$$+ \left\{ \frac{[(b_2 - a_1)(b_1 + b_2) + a_2 b_2]r}{(a_2 + b_2 - a_1)(a_2 + b_2 + b_1)} \right.$$

$$+ \frac{a_1 a_2 p}{(a_2 + b_2 - a_1)(a_2 + b_2 + b_1)}$$

$$\left. - \frac{a_2 q}{(a_2 + b_2 + b_1)} \right\} \exp\{- (a_2 + b_2)\tau\}$$

$$+\left\{ \frac{a_1 a_2 r}{(a_1+b_1)(a_2+b_2+b_1)} - \frac{[(b_1+b_2)(a_1+b_1) + a_2 b_1]}{(a_1+b_1)(a_2+b_2+b_1)} - \frac{(b_1+b_2)q}{(a_2+b_2+b_1)} \right\}$$

If $\tau_1 < T$, then for $\tau_1 \leq \tau \leq \tau_2$

$$S(\tau,1,1,1) = \frac{a_1(b_2 r - a_2 q)}{(a_1+b_1-a_2)(a_2+b_2)} \exp\{-(a_2+b_2)\tau\}$$

$$- \exp\{(a_1+b_1)(\tau_1-\tau) - a_2\tau_1 - b_2\tau\}$$

$$+\left\{ S_0 - \frac{a_1 a_2 r}{(a_1+b_1+b_2)(a_2+b_2)} + \frac{(b_1+b_2)p}{(a_1+b_1+b_2)} + \frac{[(b_1+b_2)(a_2+b_2)+a_1 b_2]q}{(a_1+b_1+b_2)(a_2+b_2)} \right\} \exp\{(a_1+b_1+b_2)(\tau_1-\tau)\}$$

$$+\left\{ \frac{a_1 a_2 r}{(a_1+b_1+b_2)(a_2+b_2)} - \frac{(b_1+b_2)p}{(a_1+b_1+b_2)} - \frac{[(b_1+b_2)(a_2+b_2)+(a_1 b_2)]q}{(a_1+b_1+b_2)(a_2+b_2)} \right\}$$

$S_0 = S(\tau_1,1,1,1)$ where τ_2 is the time fire is shifted back to X_2.

With $S(\tau,1,1,1)$ known, there is no mathematical reason why Eq. 11 cannot be solved for $S(\tau,2,1,1)$, $S(\tau,1,2,1)$, etc. in hopes that eventually a general solution for $S(\tau,x_1,x_2,y)$ might emerge. For the stochastic model corresponding to the linear law attrition process, such a general solution has been obtained by Clark [16].

No general solution for this model is known. From the complexity of the expression for $S(\tau,1,1,1)$, it is obvious that extensions will quickly become very unwieldy. A numerical approximation for $S(\tau,x_1,x_2,y)$ appears to be a more profitable line of approach.

C. NUMERICAL APPROXIMATION OF THE STOCHASTIC MODEL

In the development of the differential difference equation for $S(\tau,x_1,x_2,y)$, a suitable numerical approximation was also developed.

$$S(\tau+\Delta\tau,x_1,x_2,y) = \max_{0\le\phi\le 1} \Big\{ (1 - \{\phi a_1 y + (1-\phi)a_2 y$$

$$+ b_1x_1 + b_2x_2\}\Delta\tau)S(\tau,x_1,x_2,y) + \phi a_1 y\Delta\tau S(\tau,x_1-1,x_2,y)$$

$$+ (1-\phi)a_2 y\Delta\tau S(\tau,x_1,x_2-1,y)$$

$$+ (b_1x_1 + b_2x_2)\Delta\tau S(\tau,x_1,x_2,y-1) \Big\} \tag{11}$$

with ϕ determined by the value of the switching function $w(\tau)$ in Eq. 13.

This is essentially the Euler method of numerical approximation. It has the advantage that as $\Delta\tau\to 0$, the value computed above will converge to the true value of $S(\tau,x_1,x_2,y)$.

To obtain a reasonable approximation, the value of the time step, $\Delta\tau$, must be small. There is also an additional restriction on $\Delta\tau$. In order for the probability statements assumed in A9 to be meaningful, $\Delta\tau$ must be such that

$$a_1 y \Delta\tau \leq 1$$

$$a_2 y \Delta\tau \leq 1$$

$$(b_1 x_1 + b_2 x_2)\Delta\tau \leq 1$$

for all values of x_1, x_2 and y considered.

There are several special cases of Eq. 11.

$$S(0,x_1,x_2,y) = ry - px_1 - qx_2$$

$$S(\tau,x_1,x_2,0) = - px_1 - qx_2$$

$$S(\tau,0,0,y) = ry$$

$$S(\tau+\Delta\tau,0,x_2,y) = S(\tau,0,x_2,y)$$

$$+ \Delta\tau(b_2 x_2\{S(\tau,0,x_2,y-1) - S(\tau,0,x_2,y)$$

since $\phi = 1$ in this case and

$$S(\tau+\Delta\tau,x_1,0,y) = S(\tau,x_1,0,y)$$

$$+ \Delta\tau(b_1 x_1\{S(\tau,x_1,0,y-1) - S(\tau,x_1,0,y)\}$$

$$+ a_1 y\{S(\tau,x_1-1,0,y) - S(\tau,x_1,0,y)\})$$

since $\phi = 0$ in this case.

The above equations were programmed in Fortran IV G and run on an IBM 360/67. The output is a table which gives S as a function of τ for fixed values of x_1, x_2, and y.

In the special case of $S(\tau,1,1,1)$, the accuracy of the approximation was checked against the exact solution

developed in Section III B. The results of one such comparison are given in Table IV . It can be seen that the approximation agrees with the exact solution to the fourth decimal place.

For a given force level, the approximation must first calculate the values of S for all combinations of smaller force levels. For larger force levels, this requires a large amount of computer time and memory. For a relatively large time step of $\Delta\tau = .01$, computation of $S(\tau,5,5,5)$ for $0 \leq \tau \leq 50$ required about four minutes and 200 K of core. To solve the same problem for $S(\tau,9,9,9)$ would require about one hour and 500 K. Therefore, the numerical results from this particular approach are limited to relatively small force levels.

Table IV. Comparison of Exact Solution and Numerical Approximation

τ in minutes	Exact $S(\tau,1,1,1)$	Approximation
0	-.2000	-.2000
1	-.2018	-.2018
2	-.2037	-.2037
3	-.2055	-.2055
4	-.2073	-.2073
5	-.2091	-.2091
6	-.2108	-.2108
7	-.2126	-.2126
8	-.2143	-.2143
9	-.2161	-.2161
10	-.2178	-.2178
11	-.2195	-.2195
12	-.2212	-.2211
13	-.2229	-.2228
14	-.2245	-.2245
15	-.2262	-.2261
16	-.2279	-.2278
17	-.2295	-.2294
18	-.2311	-.2310
19	-.2327	-.2326
20	-.2343	-.2342
21	-.2358	-.2357
22	-.2374	-.2373

Table IV. (Continued)

τ in minutes	Exact $S(\tau,1,1,1)$	Approximation
23	-.2390	-.2388
24	-.2405	-.2404
25	-.2420	-.2419
26	-.2435	-.2434
27	-.2450	-.2449
28	-.2465	-.2464
29	-.2480	-.2479
30	-.2495	-.2493
31	-.2509	-.2508
32	-.2524	-.2522
33	-.2538	-.2537
34	-.2552	-.2551
35	-.2566	-.2565
36	-.2580	-.2579
37	-.2594	-.2593
38	-.2608	-.2606
39	-.2622	-.2620
40	-.2635	-.2633
41	-.2649	-.2647
42	-.2662	-.2660
43	-.2675	-.2673
44	-.2688	-.2686
45	-.2701	-.2699
46	-.2714	-.2712

Table IV. (Continued)

τ in minutes	Exact $S(\tau,1,1,1)$	Approximation
47	-.2727	-.2725
48	-.2739	-.2737
49	-.2752	-.2750
50	-.2764	-.2762

$a_1 = .005$ $a_2 = .003$ $b_1 = .007$ $b_2 = .001$

$p = .15$ $q = .45$ $r = .40$ $\Delta\tau = .01$

V. COMPARISON OF RESULTS

For the 125 combinations of force levels from $x_1 = x_2 = y = 1$ up to and including $x_1 = x_2 = y = 5$, calculations were carried out for each of the four sets of parameter values listed in Table V. A time step of .01 minutes was used in the numerical approximation. Comparisons of selected results are given in Tables VI through XIV.

These tables were constructed by extracting the force level time history from the deterministic model. Since the deterministic model calculates force levels as continuous functions of time, there is a question as to exactly when a casualty occurs. For these tables, deterministic force levels are rounded up to the next integer, i.e. $x_1 = 3.001$ is rounded to $x_1 = 4$. With these values of time and force levels, the optimal expected value function and the value of ϕ^* were obtained from the appropriate stochastic table. All comparisons are for the case $a_1p<a_2q$.

In the case $a_1p \geq a_2q$, calculations were carried for the above force level combinations with parameter set 1. In both the stochastic and deterministic model, there is no shift of fire unless x_1 is driven to zero.

Table V. Parameter Values for

Numerical Calculations

Set	a_1	a_2	b_1	b_2	p	q	r	T
1	.005	.003	.007	.001	.15	.45	.4	50
2	.005	.003	.007	.001	.15	.45	.4	100
3	.025	.015	.035	.005	.75	2.25	2.0	50
4	.05	.03	.07	.01	1.5	4.5	4.0	50

Table VI. Comparison for $x_1(0) = x_2(0) = y(0) = 1$

with Parameter Set 2

Time in minutes	Force Level x_1	x_2	y	Deterministic	Stochastic	$S(t,x_1,x_2,y)$
0	1	1	1	1	1	-.3175
49.94	1	1	1	1	0	-.2762
56.38	1	1	1	0	0	-.2686
100	1	1	1	0	0	-.2000

Deterministic Pay-off -.4127

Terminal State S1

Table VII. Comparison for $x_1(0) = x_2(0) = y(0) = 5$

with Parameter Set 2

Time in minutes	Force Level x_1	x_2	y	Deterministic	Stochastic	$S(t,x_1,x_2,y)$
0	5	5	5	1	1	-.6231
27	5	5	4	1	1	-2.1729
50	4	5	4	1	1	-1.6712
55	4	5	4	1	0	-1.6373
56	4	5	3	1	0	-2.0558
56.38	4	5	3	0	0	-2.0501
87	4	5	2	0	0	-2.1846
100	4	5	2	-	-	-2.0500

Deterministic Pay-off -2.0637

Terminal State S1

Table VIII. Comparison for $x_1(0) = 4$ $x_2(0) = 5$ $y(0) = 5$

with Parameter Set 4

Time in Minutes	Force Level x_1	x_2	y	ϕ^* Deterministic	Stochastic	$S(t,x_1,x_2,y)$
0	4	5	5	0	1	-16.9539
4	4	5	4	0	1	-20.5330
7	4	5	3	0	1	-23.0807
9	4	5	2	0	0	-25.3355
10	4	4	2	0	0	-20.2005
13	4	4	1	0	0	-20.9796
15.51	4	4	0	-	-	-24.0000

Deterministic Pay-off -23.2951

Terminal State S4

Table IX. Comparison for $x_1(0) = 2$ $x_2(0) = 5$ $y(0) = 3$

with Parameter Set 3

Time in Minutes	Force Level x_1	x_2	y	ϕ^* Deterministic	Stochastic	$S(t,x_1,x_2,y)$
0	2	5	3	1	1	-8.9286
13	2	5	2	1	1	-11.1566
18	1	5	2	1	1	-9.1233
31	1	5	1	1	1	-10.9620
35.39	1	5	1	1	0	-10.7862
41.28	1	5	1	0	0	-10.5445
50	1	5	1	-	-	-10.0000

Deterministic Pay-off -10.9471

Terminal State S1

Table X. Comparison for $x_1(0) = 4$ $x_2(0) = 5$ $y(0) = 5$

with Parameter Set 2

Time in Minutes	Force Level x_1	x_2	y	ϕ^* Deterministic	Stochastic	$S(t,x_1,x_2,y)$
0	3	5	4	1	1	-8.8300
9	3	5	3	1	1	-11.2142
13	2	5	3	1	1	-8.9147
20	2	5	2	1	1	-11.0357
34	2	5	1	1	0	-12.0296
36	1	5	1	1	0	-10.8092
41.276	1	5	1	0	0	-10.5445
50	1	5	1	-	-	-10.0000

Deterministic Pay-off -11.6727

Terminal State S1

Table XI. Comparison for $x_1(0) = 4$ $x_2(0) = 5$ $y(0) = 5$

with Parameter Set 2

Time in Minutes	Force Level x_1	x_2	y	ϕ^* Deterministic	Stochastic	$S(t,x_1,x_2,y)$
0	4	5	5	0	1	-8.4801
7	4	5	4	0	1	-10.4644
13	4	5	3	0	1	-11.7839
19	4	5	2	0	0	-12.6693
20	4	4	2	0	0	-10.3609
25	4	4	1	0	0	-11.1032
31.03	4	4	0	-	-	-13

Deterministic Pay-off -11.6476

Terminal State S4

Table XII. Comparison for $x_1(0) = 4$ $x_2(0) = 4$ $y(0) = 5$

with Parameter Set 2

Time in Minutes	Force Level x_1	x_2	y	ϕ^* Deterministic	Stochastic	$S(t,x_1,x_2,y)$
0	4	4	5	1	1	-6.2034
7	4	4	4	1	1	-8.8186
10	3	4	4	1	1	-6.225
16	3	4	3	1	1	-8.655
23	2	4	3	1	1	-6.188
26	2	4	2	1	1	-8.475
39	2	4	1	1	0	-9.517
41.28	2	4	1	0	0	-9.400
50	2	4	1	-	-	-8.500

Deterministic Pay-off -9.2835

Terminal State S1

Table XIII. Comparison for $x_1(0) = 3$ $x_2(0) = 5$ $y(0) = 4$ with Parameter Set 2

Time in Minutes	Force Level x_1	x_2	y	ϕ^* Deterministic	Stochastic	$S(t,x_1,x_2,y)$
0	3	5	4	1	1	-8.1836
9	3	5	3	1	1	-11.214
13	2	5	3	1	1	-8.918
20	2	5	2	1	1	-11.036
34	2	5	1	1	0	-12.030
36	1	5	1	1	0	-10.809
41.28	1	5	1	0	0	-10.545
50	1	5	1	-	-	-10.000

Deterministic Pay-off -11.6727

Terminal State S1

Table XIV. Comparison for $x_1(0) = x_2(0) = y(0) = 5$ with Parameter Set 1

Time in Minutes	Force Level x_1	x_2	y	ϕ^* Deterministic	Stochastic	$S(t,x_1,x_2,y)$
0	5	5	5	1	1	-1.509
5.61	5	5	5	1	0	-1.446
6.38	5	5	5	0	0	-1.435
26	5	5	4	0	0	-1.665
50	5	5	4	-	-	-1.400

Deterministic Pay-off -1.5216

Terminal State S1

VI. CONCLUSIONS

Due to the restrictions of the stochastic model to small force levels because of computer constraints, it is difficult to reach any general conclusions. There are, however, several interesting things which can be observed from the data.

In the case where $a_2q>a_1p$, the models lead to target selection rules where the time fire is shifted differs by as much as seven minutes. If the deterministic rule is to shift fire before X_1 is destroyed, the stochastic rule is to shift fire at an earlier time. This can be explained by an interpretation of the model parameters similar to that given in Section III.

The return obtained by firing at X_1 is mainly from destroying his kill capability. More Y's survive for a longer period of time. As the battle progresses, the utility of Y survivors in the future decreases until a point is reached where the return from firing at X_1 drops off rapidly. Conversely, the return from firing at X_2 is chiefly from destroying his value. As the battle progresses, the return from firing at X_2 eventually exceeds that of firing at X_1. In the deterministic model, this crossover point is known precisely. On the other hand, this crossover point is not precisely known in the stochastic model. The use of expected values tends to force an earlier shift

to keep away from the area where the X_1 return decreases rapidly. In other words, some of the future value of Y survivors is sacrificed to ensure that the higher return from firing at X_2 is not missed. In the case where $a_1 p \geq a_2 q$, this does not occur since it is always to Y's advantage to fire at X_1, both in terms of value destroyed and future survivors.

The large jump in the optimal expected value function as casualties occur is caused by the integer constraints on the stochastic model. This does not happen in the deterministic model since force levels are calculated as continuous functions. As $t \to T$, the optimal expected value function approaches the deterministic pay-off.

There are a number of areas which need additional work. Most obvious is the need for better numerical techniques for the stochastic model. One alternative is to reprogram the model to use tape storage rather than core. In this way, a series of small computer runs could be used to build up a file of values for smaller force levels which could then be accessed to compute still higher force levels. In this case, an investigation of the truncation errors would have to be made to determine the degree of accuracy in the figures for higher force levels.

Another alternative is to consider a diffusion approximation to the stochastic process. Some work in this area has been done by Taylor [10], leading to a parabolic partial differential equation which is not easily solved

analytically. It can be solved numerically. This approach appears promising since it would not be necessary to solve for all combinations of force levels below the desired ones.

This approach to the stochastic model yields no information about the time history of the average force levels or the probability of winning. Clark [16] was able to obtain expressions for the average force levels in the linear law case. If a similar solution for the square law case could be obtained, more meaningful comparisons could be made.

There are also a number of extensions to the deterministic model. The zero-one nature of the solution is hardly realistic. It is a result of the square law attrition mechanism and the constant attrition coefficients. One area of investigation would be to make the attrition coefficients a function of the fire being received. Other areas of extension are the two-on-two, one-on-m, and m-on-n battles with various types of attrition mechanisms and coefficients.

In summary, the numerical results, while inconclusive, do indicate a significant difference in results between the stochastic and deterministic modelling approaches.

APPENDIX A.

Domains of Controllability

Case I. $a_1 p \geq a_2 q$

$$S1 \quad \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y\ (T) > 0 \end{cases} \qquad \phi^*(t) = 1 \qquad o \leq t \leq T$$

Requires $T < TS1$ where

$$TS1 = \begin{cases} \dfrac{1}{\sqrt{a_1 b_1}} \ln\{\gamma\} & \text{if } a_1 b_1 y^2(0) > s^2 \\ \dfrac{1}{\sqrt{a_1 b_1}} \ln\left\{\dfrac{s}{b_2 x_2(0)}\right\} & \text{if } a_1 b_1 y^2(0) = s^2 \\ \dfrac{1}{\sqrt{a_1 b_1}} \ln\{-\gamma\} & \text{if } s^2 - \left(b_2 x_2(0)\right)^2 - a_1 b_1 y^2(0) < s^2 \\ \dfrac{1}{\sqrt{a_1 b_1}} \tanh^{-1}\left\{\dfrac{\sqrt{a_1 b_1}\ y(0)}{s}\right\} \end{cases}$$

$$\text{if } a_1 b_1 y^2(0) < s_2 - \left(b_2 x_2(0)\right)^2$$

$$\gamma = \frac{\sqrt{a_1 b_1 y^2(0) - s_2 + \left(b_2 x_2(0)\right)^2} - b_2 x_2(0)}{\sqrt{a_1 b_1}\ y(0) - s}$$

$$S2 \quad \begin{cases} x_1(TS1) = 0 \\ x_2(T) > 0 \\ y(t) >> 0 \end{cases} \qquad \phi^*(t) = \begin{cases} 1 & 0 \leq t \leq TS1 \\ TS1 < t \leq T \end{cases}$$

Requires $a_1 b_1 y^2(0) > s_2 - \left(b_2 x_2(0)\right)^2$

$TS1 \leq T$

$$S3 \quad \begin{cases} x_1(T) \quad 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases} \qquad \text{No extremals lead to this state.}$$

$$S4 \quad \begin{cases} x_1(T1) > 0 \\ x_2(T1) > 0 \\ y(T1) = 0 \end{cases} \qquad \phi^*(t) = 1 \qquad 0 \leq t \leq T1$$

Requires $a_1 b_1 y^2(0) < S - \left(b_2 x_2(0)\right)^2$

$T1 \leq T$

$$S5 \quad \begin{cases} x_1(TS1) = 0 \\ x_2(T1) >> 0 \\ y(T1) = 0 \end{cases} \qquad \phi^*(t) = \begin{cases} 1 & 0 \leq t \leq TS1 \\ 0 & TS1 < t \leq T1 \end{cases}$$

Requires $S^2 - \left(b_2 x_2(0)\right)^2 - a_1 b_1 y^2(0) < s^2$

$+ (R-1)\left(b_2 x_2(0)\right)^2$

$T1 < T$

$$S6 \quad \begin{cases} x_1(T1) > 0 \\ x_2(T1) = 0 \\ y(T1) = 0 \end{cases} \qquad \text{No extremals lead to this state.}$$

$$S7 \quad \begin{cases} x_1(TS1) = 0 \\ x_2(T1) = 0 \\ y(T1) > 0 \end{cases} \qquad \phi^*(t) = \begin{cases} 1 & 0 \leq t \leq TS1 \\ 0 & TS1 < t \leq T1 \end{cases}$$

Requires $a_1 b_1 y^2(0) > s^2 + (R-1)\left(b_2 x_2(0)\right)^2$

$T1 \leq T$

$$S8 \begin{cases} x_1(T1) = 0 \\ x_2(T1) = 0 \\ y(T1) > 0 \end{cases} \qquad \text{No extremals lead to this state.}$$

<u>Case II</u>. $a_1 p < a_2 q$

$$S1 \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) > 0 \end{cases} \qquad \phi^*(t) = \begin{cases} 0 & 0 \leq t \leq T \quad \text{if } \tau S1 \geq T \\ 1 & 0 \leq t \leq T - \tau S1 \\ 0 & T - \tau S1 < t \leq T \text{ if } \tau S1 < T \end{cases}$$

$$\text{where } \tau S1 = \frac{1}{\sqrt{a_2 b_2}} \ln \left\{ \frac{Z + \sqrt{Z^2 + \gamma^2 - 1}}{1 + \gamma} \right\}$$

Requires $T < TS1$ where

$$TS1 = \begin{cases} \dfrac{1}{\sqrt{a_2 b_2}} \tanh^{-1} \left\{ \dfrac{\sqrt{a_1 b_1}\; y(0)}{s\sqrt{R}} \right\} & \text{if } a_1 b_1 y(0)^2 < R\{s^2 - (b_1 x_1(0))^2\} \\ \dfrac{1}{\sqrt{a_2 b_2}} \ln\{\gamma\} & \text{if } R\{s^2 - (b_1 x_1(0))^2\} \leq a_1 b_1 y^2(0) \leq Rs^2 \\ \dfrac{1}{\sqrt{a_2 b_2}} \ln \left\{ \dfrac{s}{b_2 x_2(0)} \right\} & \text{if } a_1 b_1 y^2(0) = Rs^2 \\ \dfrac{1}{\sqrt{a_2 b_2}} \ln\{-\gamma\} & \text{if } Rs^2 < a_1 b_1 y^2(0) \end{cases}$$

$$\gamma = \frac{b_1 x_1(0)\sqrt{R} - \sqrt{a_1 b_1 y^2(0) - R\{s^2 - (b_1 x_1(0))^2\}}}{s\sqrt{R} - \sqrt{a_1 b_1}\, y(0)}$$

$$S2 \quad \begin{cases} x_1(TS2) > 0 \\ x_2(T) > 0 \\ y(T) > 0 \end{cases} \qquad \phi^*(t) = \begin{cases} 1 & 0 \leq t \leq TS2 \\ 0 & TS2 < t \leq T1 \end{cases}$$

Requires $a_1 b_1 y^2(0) > s^2 - (b_2 x_2(0))^2$ and

$\tau S1 + TS2 < T$ where

$$TS2 = \begin{cases} \frac{1}{\sqrt{a_1 b_1}} \ln\{\gamma\} & \text{if } a_1 b_1 y^2(0) > s^2 \\ \frac{1}{\sqrt{a_1 b_1}} \ln\left\{\frac{s}{b_2 x_2(0)}\right\} & a_1 b_1 y^2(0) = s^2 \\ \frac{1}{\sqrt{a_1 b_1}} \ln\{-\gamma\} & \text{if } a_1 b_1 y^2(0) < s^2 \end{cases}$$

$$\gamma = \frac{\sqrt{a_1 b_1 y^2(0) - s^2 + (b_2 x_2(0))^2} - b_2 x_2(0)}{\sqrt{a_1 b_1}\, y(0) - s}$$

$$S3 \quad \begin{cases} x_1(TS3) > 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases} \qquad \phi^*(t) = \begin{cases} 1 & 0 \leq t \leq TS3 \\ 0 & TS3 < t \leq T \end{cases}$$

Requires $TS1 \leq T$ and $T - \tau S1 \leq TS3$ where $TS3$ is a root of

$$0 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{R}}\right)[s \cosh(\gamma_1) - \sqrt{a_1 b_1}\, y(0) \sinh(\gamma_1)]$$

$$+ \frac{1}{2}\left(1 - \frac{1}{\sqrt{R}}\right)\left[s \cosh(\gamma_2) - \sqrt{a_1 b_1}\; y(0) \sinh(\gamma_2)\right]$$

$$- s \cosh\left(\sqrt{a_1 b_1}\; TS3\right)$$

$$+ \sqrt{a_1 b_1}\; y(0) \sinh\left(\sqrt{a_1 b_1}\; TS3\right) + b_2 x_2(0)$$

$$\gamma_1 = \sqrt{a_2 b_2}\,(T - TS3) + \sqrt{a_1 b_1}\; TS3$$

$$\gamma_2 = \sqrt{a_2 b_2}\,(T - TS3) - \sqrt{a_1 b_1}\; TS3$$

$$S4 \quad \begin{cases} x_1(T1) > 0 \\ x_2(T1) > 0 \\ y(T1) = 0 \end{cases}$$

$$\tau S4 = \frac{1}{\sqrt{a_2 b_2}} \ln \left\{ Z + \sqrt{Z^2 - 1} \right\}$$

(a.) When $\tau S4 > T1 \qquad \phi^*(t) = 0 \qquad 0 \leq t \leq T1$

Requires

$$a_1 b_1 y^2(0) \leq R s^2 \left\{ 1 - \frac{1}{Z^2} \right\}$$

$$a_1 b_1 y^2(0) < R \left\{ s^2 - \left(b_1 x_1(0)\right)^2 \right\}$$

$$a_1 b_1 y^2(0) < s^2 + (R - 1)\left(b_2 x_2(0)\right)^2$$

$$T1 \leq T$$

(b.) When $\tau S4 < T1$

(1.) When $R - \sqrt{R(R - 1)} < \delta < 1$

$$\phi^*(t) = \begin{cases} 1 & 0 \leq t \leq T1 - \tau S4 \\ 0 & T1 - \tau S4 \leq t \leq T1 \end{cases}$$

Requires

$$a_1 b_1 y^2(0) > s_2 + A\left(b_2 x_2(0)\right)^2$$

$$a_1 b_1 y^2(0) < s^2 + B\left(b_2 x_2(0)\right)^2$$

$$a_1 b_1 y^2(0) > Rs^2 \left\{ 1 - \frac{1}{Z^2} \right\}$$

$$T1 \leq T$$

(2.) When $\delta = R - \sqrt{R(R-1)}$

$$\phi^*(t) = \begin{cases} 1 & 0 \leq t \leq TS4 \\ 0 & TS4 < t \leq T1 \end{cases}$$

Requires

$$a_1 b_1 y^2(0) = s^2$$

$$b_2 x_2(0) > b_1 x_1(0) \left\{ \sqrt{\frac{R}{R-1}} - 1 \right\}$$

$$T1 \leq T$$

where TS4 is any value less than TS2

$$S5 \quad \begin{cases} x_1(TS2) = 0 \\ x_2(T1) > 0 \\ y(T1) = 0 \end{cases} \qquad \phi^*(t) = \begin{cases} 1 & 0 \leq t \leq TS6 \\ 0 & TS6 < t \leq T1 \end{cases}$$

Requires

$$a_1 b_1 y^2(0) \geq s^2 + B\left(b_2 x_2(0)\right)^2$$

$$a_1 b_1 y^2(0) < s^2 + (R-1)\left(b_2 x_2(0)\right)^2$$

$$T1 \leq T$$

$$S6 \quad \begin{cases} x_1(T1) > 0 \\ x_2(T1) = 0 \\ y(T1) = 0 \end{cases} \qquad \phi^*(t) = \begin{cases} 1 & 0 \leq t \leq TS6 \\ 0 & TS6 < t \leq T1 \end{cases}$$

Requires

$$a_1 b_1 y^2 \leq s^2 + A\left(b_2 x_2(0)\right)^2$$

$$a_1 b_1 y^2(0) \geq R\{s^2 - \left(b_1 x_1(0)\right)^2\}$$

$$a_1 b_1 y^2(0) < s^2 + (R-1)\left(b_2 x_2(0)\right)^2$$

$$T1 \leq T$$

where

$$TS6 = \begin{cases} \dfrac{1}{\sqrt{a_1 b_1}} \ln(\gamma) & \text{if } a_1 b_1 y^2(0) > s^2 \\ \dfrac{1}{\sqrt{a_1 b_1}} \ln\left\{\dfrac{y(0)}{y(TS6)}\right\} & \text{if } a_1 b_1 y^2(0) = s^2 \\ \dfrac{1}{\sqrt{a_1 b_1}} \ln(-\gamma) & \text{if } a_1 b_1 y^2(0) < s^2 \end{cases}$$

where

$$\gamma = \frac{y(TS6) - \sqrt{y^2(TS6) - y^2(0) + \dfrac{s^2}{a_1 b_1}}}{y(0) - \dfrac{s}{\sqrt{a_1 b_1}}}$$

and

$$y(TS6) = \sqrt{\frac{x_2(0)}{a_2}\left\{(2R-1) b_2 x_2(0) + 2\sqrt{s^2 + R(R-1)\left(b_2 x_2(0)\right)^2 - a_1 b_1 y(0)^2}\right\}}$$

S7 $\begin{cases} x_1(T1) = 0 \\ x_2(T1) = 0 \\ y(T1) > 0 \end{cases}$ $\phi^*(t) = \begin{cases} 1 & 0 \le t \le TS2 \\ 0 & TS2 < t \le T1 \end{cases}$

Requires

$$a_1 b_1 y^2(0) > s^2 + (R-1)\left(b_2 x_2(0)\right)^2$$

$$T1 \le T$$

S8 $\begin{cases} x_1(T1) = 0 \\ x_2(T1) = 0 \\ y(T1) > 0 \end{cases}$ No extremals lead to this state.

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ABSTRACT

The simplest target selection problem in heterogeneous combat is the two-on-one battle. For a prescribed duration battle, deterministic and stochastic models using Lanchester's square law attrition mechanism are developed. Solutions of these models, obtained by the application of optimal control theory, are given, including the complete solution of the deterministic model and the optimal target selection rule and expected pay-off for special cases of the stochastic model. A numerical approximation for the general case of the stochastic model is obtained. Comparison of numerical results shows that the target selection rule specified by the deterministic models differs from the one given by the stochastic model.

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